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ORIGINAL PAPER

A note on the generalized Rayleigh equation: limit cycles and stability

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Abstract The aim of the present paper is to give an analytical proof on the existence and stability of the limit cycles in the generalized Rayleigh equation, which models diabetic chemical processes through a constant area duct where the effect of heat addition or rejection is considered, $\frac{d^2x}{dt^2} + x = \varepsilon \left(1 - (\frac{dx}{dt})^{2n}\right) \frac{dx}{dt}$ where n is a positive integer and ε a small real parameter. The main tool used for it is the averaging theory.

Keywords Generalized Rayleigh equation · Limit cycle · Averaging theory

1 Introduction

The Rayleigh equation is a continuous dynamical system given by the oscillator $\frac{d^2x}{dt^2} + x = \varepsilon \left(1 - (\frac{dx}{dt})^2\right) \frac{dx}{dt}$ and models a batch distillation process where a fixed quantity of feed is charged to the distillation column and separation of volatile products is done by supplying heat through the still or reboiler. As the boiling point of liquid is reached a portion of liquid starts to vaporize to the top of the column and directed to the condenser, the vapor leaving the column is always at equilibrium with the liquid in the bottom of the column, thus a material balance equation is produced by Rayleigh called the Rayleigh Equation which provides standard equilibrium data. See for more details on this equation [1]. Our aim on the present paper is to consider an oscillator which generalizes the Rayleigh equation

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$$\frac{d^2x}{dt^2} + x = \varepsilon\left(1 - \left(\frac{dx}{dt}\right)^{2n}\right)\frac{dx}{dt} \tag{1}$$

and using averaging theory give information on its limit cycles and stability. The structure of this note is the following. In Sect. 2 we present the auxiliary results coming from the averaging theory and in Sect. 3 we present our main results on the generalized Rayleigh equation.

2 Auxiliary results from the averaging theory

We present in this section a basic result known as *first order averaging theorem*. For a general introduction to averaging theory see for instance the book of Verhulst [2].

We consider the differential equation

$$\frac{dX}{dt} = \varepsilon F_1(t, X) + \varepsilon^2 R(t, X, \varepsilon) \tag{2}$$

where the smooth functions $F_1 : \mathbb{R} \times U \to \mathbb{R}^n$ and $R : \mathbb{R} \times U \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are T-periodic in the first variable and U is an open subset of $\mathbb{R}^n$.

We define the averaged system associated with system (2) as

$$\frac{dZ}{dt} = \varepsilon f_1(Z) \tag{3}$$

where $f_1 : U \to \mathbb{R}^n$ is given by

$$f_1(Z) = \frac{1}{T}\int_0^T F_1(s, Z)ds. \tag{4}$$

We present, in the following result, the conditions for the singularities of system (3) which are associated with the periodic solutions of the differential system (2).

Theorem 1 *Assume that for $a \in U$ with $f_1(a) = 0$, there exists a neighborhood V of a such that $f_1(Z) \neq 0$ for all $Z \in \bar{V} - \{a\}$ and $\det(df_1(a)) \neq 0$. Then, for $|\varepsilon| > 0$ sufficiently small, there exist a T-periodic solution $X(t, \varepsilon)$ of the system (2) such that $X(0, \varepsilon) \to a$ as $\varepsilon \to 0$.*

Theorem 1 can be announced with weaker hypothesis using Brouwer degree. For a proof of Theorem 1 see [3].

For a good use of the first order averaging theory the second order differential Eq. (1) is transformed in the following first order planar system

$$\begin{cases} \dfrac{dX}{dt} = Y \\ \dfrac{dY}{dt} = -X + \varepsilon(1 - Y^{2n})Y \end{cases}. \tag{5}$$

Introducing polar coordinates $X = r\cos\theta$, $Y = r\sin\theta$ we obtain

$$\begin{cases} \dfrac{dr}{dt} = \varepsilon r \sin^2\theta(1 - r^{2n}\sin^{2n}\theta) \\ \dfrac{d\theta}{dt} = -1 + \varepsilon\cos\theta\sin\theta(1 - r^{2n}\sin^{2n}\theta) \end{cases}.$$

On the other hand the previous system, after a change of variable, is reduced to following differential equation

$$\frac{dr}{d\theta} = -\varepsilon r \sin^2\theta(1 - r^{2n}\sin^{2n}\theta) - \varepsilon^2 R(r, \theta, \varepsilon) \tag{6}$$

which is in a normal form format allowing to apply Theorem 1.

3 Main results on the generalized Rayleigh equation

Theorem 2 *The second order differential equation*

$$\frac{d^2x}{dt^2} + x = \varepsilon\left(1 - \left(\frac{dx}{dt}\right)^{2n}\right)\frac{dx}{dt}$$

with $\varepsilon << 1$ *and* $n \in \mathbb{N}$ *has a unique stable limit cycle given by*

$$r_n(t) = \alpha_n + O(\varepsilon),\ \theta(t) = t + O(\varepsilon)$$

with

$$\alpha_n = \left((-1)^{1-n}\frac{\Gamma\left(-\frac{1}{2} - n\right)\Gamma\left(2 + n\right)}{2\sqrt{\pi}}\right)^{\frac{1}{2n}}$$

and Γ *the Gamma function. The succession* $(\alpha_n)_{n\in\mathbb{N}}$ *verifies* $\lim\limits_{n\to\infty}\alpha_n = 1$.

Proof Using the averaging theory of first order we compute

$$f_1(r) = \frac{r}{2\pi}\int\limits_0^{2\pi}\sin^2\theta(1 - r^{2n}\sin^{2n}\theta)d\theta.$$

Using the Γ function of Euler we obtain

$$f_1(r) = \frac{r}{2}\left(1 + \frac{2(-1)^n\sqrt{\pi}r^{2n}}{\Gamma\left(-\frac{1}{2} - n\right)\Gamma\left(2 + n\right)}\right).$$

The no null zeros α_n of the equation $f_1(r) = 0$ with $\frac{d}{dr}f_1(\alpha_n) \neq 0$ correspond to limit cycles of (1). We obtain

$$\alpha_n = \left((-1)^{1-n} \frac{\Gamma\left(-\frac{1}{2}-n\right)\Gamma(2+n)}{2\sqrt{\pi}} \right)^{\frac{1}{2n}}, \quad \frac{d}{dr} f_1(\alpha_n) = -n$$

and the limit cycle is stable. Using the properties of the Γ function, the asymptotic expansion to the second order in the infinitesimal $\frac{1}{n}$ of α_n are

$$\alpha_n = 1 - \frac{\log\left(\frac{4}{n\pi}\right)}{n} + o\left(\frac{1}{n^2}\right).$$

In consequence $\lim_{n\to\infty} \alpha_n = 1$. □

Using Theorem 1 we conclude immediately the following result.

Corollary 3 *The limit cycle in first approximation is given by*

$$X = (\alpha_n + O(\varepsilon)) \cos(t + O(\varepsilon)), \; Y = (\alpha_n + O(\varepsilon)) \sin(t + O(\varepsilon))$$

and for n huge

$$X = \left(1 - \frac{\log\left(\frac{4}{n\pi}\right)}{n} + o\left(\frac{1}{n^2}\right) + O(\varepsilon)\right) \cos(t + O(\varepsilon))$$
$$Y = \left(1 - \frac{\log\left(\frac{4}{n\pi}\right)}{n} + o\left(\frac{1}{n^2}\right) + O(\varepsilon)\right) \sin(t + O(\varepsilon))$$

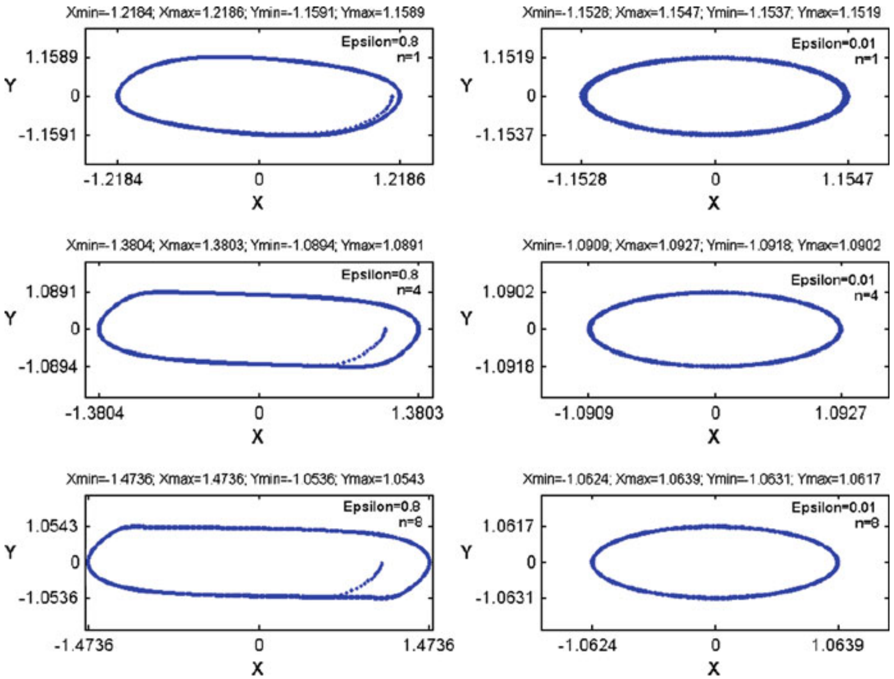

Fig. 1 Solutions of (5)

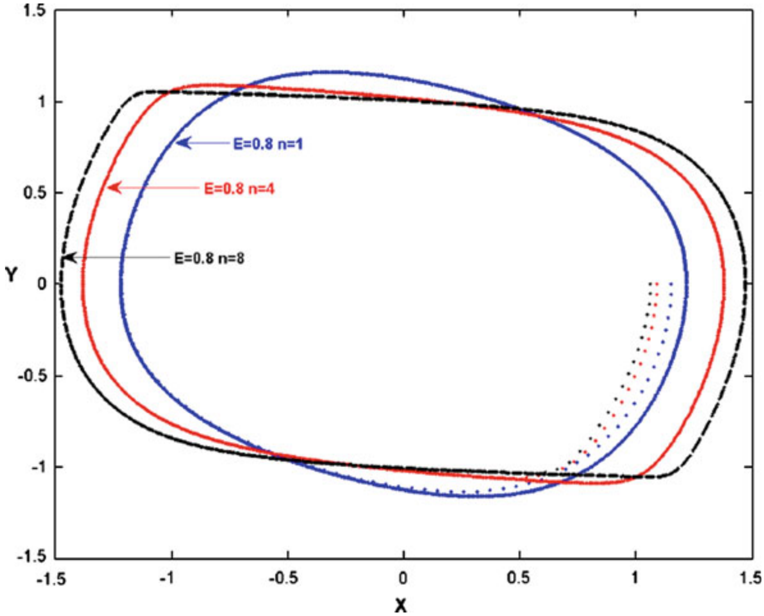

Fig. 2 Evolution of the limit cycle for $\varepsilon = 0.8$

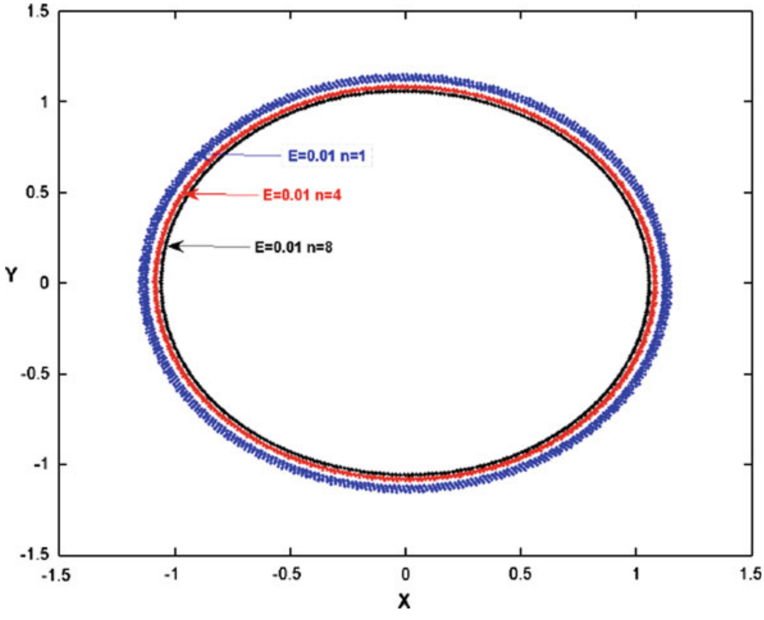

Fig. 3 Evolution of the limit cycle for $\varepsilon = 0.01$

Remark 4 Figure 1 shows the solution of the planar system (5) with $X(0) = \left((-1)^{1-n}\frac{\Gamma(-\frac{1}{2}-n)\Gamma(2+n)}{2\sqrt{\pi}}\right)^{\frac{1}{2n}}$, $Y(0) = 0$ when $\varepsilon = 0.8$ and 0.01 for $n = 1, 4$ and 8. Figures 2 and 3 show the evolution of the form of the limit cycle for the different values of n respectively for $\varepsilon = 0.8$ and 0.01.

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